Generating Functions, Weighted and Non-Weighted Sums for Powers of Second-Order Recurrence Sequences

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Abstract

In this paper we find closed forms of the generating function $\sum_{k=0}^{\infty} U_n^r x^n$, for powers of any non-degenerate second-order recurrence sequence, $U_{n+1} = aU_n + bU_{n-1},\ a^2+4b \neq 0$, completing a study began by Carlitz [1] and Riordan [4] in 1962. Moreover, we generalize a theorem of Horadam [3] on partial sums involving such sequences. Also, we find closed forms for weighted (by binomial coefficients) partial sums of powers of any non-degenerate second-order recurrence sequences. As corollaries we give some known and seemingly unknown identities and derive some very interesting congruence relations involving Fibonacci and Lucas sequences.

1 Introduction

DeMoivre (1718) used the generating function (found by using the recurrence) for the Fibonacci sequence $\sum_{i=0}^{\infty} F_i x^i = \dfrac{x}{1-x-x^2}$, to obtain the identities $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, L_n = \alpha^n + \beta^n$ (*Lucas numbers*) with $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$, called *Binet formulas*, in honor of Binet who in fact rediscovered them more than one hundred years later, in 1843 (see [6]).

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Reciprocally, using the Binet formulas, we can find the generating function easily $\sum_{i=0}^{\infty} F_i x^i = \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} (\alpha^i - \beta^i) x^i = \frac{1}{\sqrt{5}} \left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x} \right) = \frac{x}{1-x-x^2}$, since $\alpha\beta = -1, \alpha + \beta = 1$.

The question that arises is whether we can find a closed form for the generating function for powers of Fibonacci numbers, or better yet, for powers of any second-order recurrence sequences. Carlitz [1] and Riordan [4] were unable to find the closed form for the generating functions $F(r,x)$ of F_n^r, but found a recurrence relation among them, namely

$$(1 - L_r x + (-1)^r x^2) F(r,x) = 1 + rx \sum_{j=1}^{[\frac{r}{2}]} (-1)^j \frac{A_{rj}}{j} F(r-2j, (-1)^j x),$$

with A_{rj} having a complicated structure (see also [2]). We are able to complete the study began by them and find a closed form for the generating function for powers of any non-degenerate second-order recurrence sequence. We would like to point out, that this "forgotten" technique we employ can be used to attack successfully other sums or series involving any second-order recurrence sequence. In this paper we also find closed forms for non-weighted partial sums for non-degenerate second-order recurrence sequences, generalizing a theorem of Horadam [3] and also weighted (by the binomial coefficients) partial sums for such sequences.

2 Generating Functions

We consider the general non-degenerate second-order recurrences $U_{n+1} = aU_n + bU_{n-1}$, $a^2 + 4b \neq 0$. We intend to find the generating function $U(r,x) = \sum_{i=0}^{\infty} U_i^r x^i$. It is known that the Binet formula for the sequence U_n is $U_n = A\alpha^n - B\beta^n$, where $\alpha = \frac{1}{2}(a + \sqrt{a^2+4b}), \beta = \frac{1}{2}(a - \sqrt{a^2+4b})$ and $A = \frac{U_1 - U_0\beta}{\alpha-\beta}, B = \frac{U_1 - U_0\alpha}{\alpha-\beta}$. We associate the sequence $V_n = \alpha^n + \beta^n$, which satisfies the same recurrence, with the initial conditions $V_0 = 2, V_1 = a$.

Theorem 1. *We have*

$$U(r,x)=\sum_{k=0}^{\frac{r-1}{2}}(-1)^kA^kB^k\binom{r}{k}\frac{A^{r-2k}-B^{r-2k}+(-b)^k(B^{r-2k}\alpha^{r-2k}-A^{r-2k}\beta^{r-2k})x}{1-(-b)^kV_{r-2k}-x^2},$$

if r odd, and

$$\begin{aligned}U(r,x)=&\sum_{k=0}^{\frac{r}{2}-1}(-1)^kA^kB^k\binom{r}{k}\frac{B^{r-2k}+A^{r-2k}-(-b)^k(B^{r-2k}\alpha^{r-2k}+A^{r-2k}\beta^{r-2k})x}{1-(-b)^kV_{r-2k}x+x^2}\\&+\binom{r}{\frac{r}{2}}\frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-1)^{\frac{r}{2}}x},\ \textit{if } r \textit{ even.}\end{aligned}$$

Proof. We evaluate

$$\begin{aligned}U(r,x)&=\sum_{i=0}^{\infty}\left(\sum_{k=0}^{r}\binom{r}{k}(A\alpha^i)^k(-B\beta^i)^{r-k}\right)x^i\\&=\sum_{k=0}^{r}\binom{r}{k}A^k(-B)^{r-k}\sum_{i=0}^{\infty}(\alpha^k\beta^{r-k}x)^i\\&=\sum_{k=0}^{r}\binom{r}{k}A^k(-B)^{r-k}\frac{1}{1-\alpha^k\beta^{r-k}x}.\end{aligned}$$

If r odd, then associating $k\leftrightarrow r-k$, we get

$$\begin{aligned}U(r,x)&=\sum_{k=0}^{\frac{r-1}{2}}\binom{r}{k}\left(\frac{A^k(-B)^{r-k}}{1-\alpha^k\beta^{r-k}x}+\frac{A^{r-k}(-B)^k}{1-\alpha^{r-k}\beta^kx}\right)\\&=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\left(\frac{A^{r-k}B^k}{1-\alpha^{r-k}\beta^kx}-\frac{A^kB^{r-k}}{1-\alpha^k\beta^{r-k}x}\right)\\&=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\frac{A^{r-k}B^k-A^kB^{r-k}+(A^kB^{r-k}\alpha^{r-k}\beta^k-A^{r-k}B^k\alpha^k\beta^{r-k})x}{1-(\alpha^k\beta^{r-k}+\alpha^{r-k}\beta^k)x+\alpha^r\beta^rx^2}\\&=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\frac{A^{r-k}B^k-A^kB^{r-k}+(-1)^kb^k(A^kB^{r-k}\alpha^{r-2k}-A^{r-k}B^k\beta^{r-2k})x}{1-(-1)^kb^kV_{r-2k}-x^2}.\end{aligned}$$

If r even, then then associating $k\leftrightarrow r-k$, except for the middle term, we get

$$\begin{aligned}U(r,x)&=\sum_{k=0}^{\frac{r}{2}-1}\binom{r}{k}\left(\frac{A^k(-B)^{r-k}}{1-\alpha^k\beta^{r-k}x}+\frac{A^{r-k}(-B)^k}{1-\alpha^{r-k}\beta^kx}\right)+\binom{r}{\frac{r}{2}}\frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-1)^{\frac{r}{2}}x}\\&=\sum_{k=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\left(\frac{A^{r-k}B^k}{1-\alpha^{r-k}\beta^kx}+\frac{A^kB^{r-k}}{1-\alpha^k\beta^{r-k}x}\right)+\binom{r}{\frac{r}{2}}\frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-1)^{\frac{r}{2}}x}\end{aligned}$$

$$= \sum_{k=0}^{\frac{r}{2}-1} (-1)^k \binom{r}{k} \frac{A^k B^{r-k} + A^{r-k} B^k - (A^k B^{r-k} \alpha^{r-k} \beta^k + A^{r-k} B^k \alpha^k \beta^{r-k})x}{1 - (\alpha^k \beta^{r-k} + \alpha^{r-k} \beta^k)x + \alpha^r \beta^r x^2}$$
$$+ \binom{r}{\frac{r}{2}} \frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-1)^{\frac{r}{2}}x}$$
$$= \sum_{k=0}^{\frac{r}{2}-1} (-1)^k \binom{r}{k} \frac{A^k B^{r-k} + A^{r-k} B^k - (-1)^k b^k (A^k B^{r-k} \alpha^{r-2k} + A^{r-k} B^k \beta^{r-2k})x}{1 - (-1)^k b^k V_{r-2k} x + x^2}$$
$$+ \binom{r}{\frac{r}{2}} \frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-1)^{\frac{r}{2}}x}.$$

□

We can derive the following beautiful identities

Corollary 2. *If $U_0 = 0$, then $A = B = \frac{U_1}{\alpha-\beta}$ and*

$$U(r,x) = A^{r-1} \sum_{k=0}^{\frac{r-1}{2}} \binom{r}{k} \frac{b^k U_{r-2k} x}{1-(-b)^k V_{r-2k} x - x^2}, \text{ if } r \text{ odd}$$
$$U(r,x) = A^r \sum_{k=0}^{\frac{r}{2}-1} (-1)^k \binom{r}{k} \frac{2-(-b)^k V_{r-2k} x}{1-(-b)^k V_{r-2k} x + x^2} + \binom{r}{\frac{r}{2}} \frac{(-1)^{\frac{r}{2}} A^r}{1-(-1)^{\frac{r}{2}} x}, \text{ if } r \text{ even.}$$

Corollary 3. *If $\{U_n\}_n$ is a non-degenerate second-order recurrence sequence and $U_0 = 0$, then*

$$U(1,x) = \frac{A^2 U_1 x}{1 - V_1 x - x^2} \tag{1}$$
$$U(2,x) = \frac{-A^2 (V_2+2) x (x-1)}{(x+1)(x^2 - V_2 x + 1)} \tag{2}$$
$$U(3,x) = \frac{A^4 U_1 x \left((a^2+2b) - 2a^2 bx - (a^2+2b)x^2\right)}{(1 - V_3 x - x^2)(1 + bV_1 x - x^2)} \tag{3}$$

Proof. We use Corollary 2. The first two identities are straightforward. Now,

$$\begin{aligned} U(3,x) &= A^4 \left(\frac{U_3 x}{1 - V_3 x - x^2} + \frac{bU_1 x}{1 + bV_1 x - x^2} \right) \\ &= A^4 x \frac{U_3 + bU_1 + b(U_3 V_1 - U_1 V_3)x - (U_3 + bU_1)x^2}{(1 - V_3 x - x^2)(1 + bV_1 x - x^2)} \\ &= \frac{A^4 U_1 x \left((a^2+2b) - 2a^2 bx - (a^2+2b)x^2\right)}{(1 - V_3 x - x^2)(1 + bV_1 x - x^2)}, \end{aligned}$$

since $U_3 + bU_1 = (a^2 + 2b)U_1$ and $U_3V_1 - U_1V_3 = -2a^2U_1$. $\square$

Remark 4. *If U_n is the Fibonacci sequence, then $a = b = 1$, and if U_n is the Pell sequence, then $a = 2, b = 1$.*

3 Horadam's Theorem

Horadam [3] found some closed forms for partial sums $S_n = \sum_{i=1}^{n} P_i$, $S_{-n} = \sum_{i=1}^{n} P_{-i}$, where P_n is the generalized Pell sequence, $P_{n+1} = 2P_n + P_{n-1}, P_1 = p, P_2 = q$. Let p_n be the ordinary Pell sequence, with $p = 1, q = 2$, and q_n be the sequence satisfying the same recurrence, with $p = 1, q = 3$. He proved

Theorem 5 (Horadam). *For any n,*

$$\begin{aligned} S_{4n} &= q_{2n}(pq_{2n-1} + qq_{2n}) + p - q; & S_{4n-2} &= q_{2n-1}(pq_{2n-2} + qq_{2n-1}) \\ S_{4n+1} &= q_{2n}(pq_{2n} + qq_{2n+1}) - q; & S_{4n-1} &= q_{2n}(pq_{2n-2} + qq_{2n-1}) - q \\ S_{-4n} &= q_{2n}(-pq_{2n+2} + qq_{2n+1}) + 3p - q; & S_{-4n+2} &= q_{2n}(-pq_{2n} + qq_{2n-1}) + 2p \\ S_{-4n+1} &= q_{2n}(pq_{2n+1} - qq_{2n}) + p; & S_{-4n-1} &= q_{2n+1}(pq_{2n+2} - qq_{2n+1}) + 2p - q. \end{aligned}$$

We observe that Horadam's theorem is a particular case of the partial sum for a non-degenerate second-order recurrence sequence U_n. In fact, we find $S^U_{n,r}(x) = \sum_{i=0}^{n} U_i^r x^i$. For simplicity, we let $U_0 = 0$. Thus, $U_n = A(\alpha^n - \beta^n)$ and $V_n = \alpha^n + \beta^n$. We prove

Theorem 6. *We have*

$$S^U_{n,r}(x) = A^{r-1} \sum_{k=0}^{\frac{r-1}{2}} \binom{r}{k} \frac{U_{r-2k}x - (-1)^{kn} U_{(r-2k)(n+1)} x^{n+1} - (-1)^{k(n+1)} U_{(r-2k)n} x^{n+2}}{1 - (-1)^k V_{r-2k} x - x^2} \tag{4}$$

if r odd, and

$$S^U_{n,r}(x) = A^r \sum_{k=0}^{\frac{r-1}{2}} \binom{r}{k} \frac{V_{r-2k}x - (-1)^{kn} V_{(r-2k)(n+1)} x^{n+1} - (-1)^{k(n+1)} V_{(r-2k)n} x^{n+2}}{1 - (-1)^k V_{r-2k} x + x^2} \\ + A^r \binom{r}{\frac{r}{2}} \frac{(-1)^{\frac{r}{2}(n+1)} x^{n+1} - 1}{(-1)^{\frac{r}{2}} x - 1} \tag{5}$$

if r even.

Proof. We evaluate

$$\begin{aligned}
S^U_{n,r}(x) &= \sum_{i=0}^{n} \sum_{k=0}^{r} \binom{r}{k} (A\alpha)^k (-A\beta)^{r-k} x^i \\
&= A^r \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \sum_{i=0}^{n} (\alpha^k \beta^{r-k} x)^i \\
&= A^r \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \frac{(\alpha^k \beta^{r-k} x)^{n+1} - 1}{\alpha^k \beta^{r-k} x - 1}.
\end{aligned}$$

Assume r odd. Then, associating $k \leftrightarrow r-k$, we get

$$\begin{aligned}
S^U_{n,r}(x) &= A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \left(\frac{(\alpha^{r-k}\beta^k x)^{n+1} - 1}{\alpha^{r-k}\beta^k x - 1} - \frac{(\alpha^k \beta^{r-k} x)^{n+1} - 1}{\alpha^k \beta^{r-k} x - 1} \right) \\
&= A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \frac{(\alpha^k \beta^{r-k} x - 1)(\alpha^{(r-k)(n+1)} \beta^{k(n+1)} x^{n+1} - 1)}{} \\
&\qquad \frac{-(\alpha^{r-k}\beta^k x - 1)(\alpha^{k(n+1)} \beta^{(r-k)(n+1)} x^{n+1} - 1)}{(\alpha^k \beta^{r-k} x - 1)(\alpha^{r-k} \beta^k x - 1)} \\
&= A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \frac{\alpha^{k(n+1)-kn} \beta^{r+kn} x^{n+2}}{} \\
&\qquad \frac{-\alpha^{(r-k)(n+1)} \beta^{k(n+1)} x^{n+1} - \alpha^k \beta^{r-k} x}{} \\
&\qquad \frac{-\alpha^{r+kn} \beta^{r(n+1)-kn} x^{n+2} + \alpha^{r-k} \beta^k x}{} \\
&\qquad \frac{-\alpha^{k(n+1)} \beta^{(r-k)(n+1)} \beta^{(r-k)(n+1)} x^{n+1}}{1 - (-1)^k (\alpha^{r-2k} + \beta^{r-2k}) + \alpha^r \beta^r x^2}
\end{aligned}$$

$$
\begin{aligned}
&= A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \frac{(-1)^k(\alpha^{r-2k} - \beta^{r-2k})x - (-1)^{k(n+1)}(\alpha^{(r-2k)(n+1)}}{} \\
&\qquad \frac{-\beta^{(r-2k)(n+1)})x^{n+1} + (-1)^{r+kn)}(\alpha^{(r-2k)n} - \beta^{(r-2k)n})x^{n+2}}{1-(-1)^k V_{r-2k}x - x^2} \\
&= A^{r-1} \sum_{k=0}^{\frac{r-1}{2}} \binom{r}{k} \frac{U_{r-2k}x - (-1)^{kn}U_{(r-2k)(n+1)}x^{n+1} - (-1)^{k(n+1)}U_{(r-2k)n}x^{n+2}}{1-(-1)^k V_{r-2k}x - x^2}.
\end{aligned}
$$

Assume r even. Then, as before, associating $k \leftrightarrow r-k$, except for the middle term, we get

$$
\begin{aligned}
S^U_{n,r}(x) &= A^r \sum_{k=0}^{\frac{r-1}{2}} (-1)^k \binom{r}{k} \frac{(-1)^k(\alpha^{r-2k} + \beta^{r-2k})x - (-1)^{k(n+1)}(\alpha^{(r-2k)(n+1)}}{} \\
&\qquad \frac{+\beta^{(r-2k)(n+1)})x^{n+1} + (-1)^{r+kn)}(\alpha^{(r-2k)n} + \beta^{(r-2k)n})x^{n+2}}{1-(-1)^k V_{r-2k}x + x^2} \\
&\qquad + A^r \binom{r}{\frac{r}{2}} \frac{(-1)^{\frac{r}{2}(n+1)}x^{n+1} - 1}{(-1)^{\frac{r}{2}}x - 1} \\
&= A^r \sum_{k=0}^{\frac{r-1}{2}} \binom{r}{k} \frac{V_{r-2k}x - (-1)^{kn}V_{(r-2k)(n+1)}x^{n+1} - (-1)^{k(n+1)}V_{(r-2k)n}x^{n+2}}{1-(-1)^k V_{r-2k}x + x^2} \\
&\qquad + A^r \binom{r}{\frac{r}{2}} \frac{(-1)^{\frac{r}{2}(n+1)}x^{n+1} - 1}{(-1)^{\frac{r}{2}}x - 1}.
\end{aligned}
$$

□

Taking $r = 1$, we get the partial sum for any non-degenerate second-order recurrence sequence, with $U_0 = 0$,

Corollary 7. $S^U_{n,1}(x) = \dfrac{x\left(U_1 - U_{n+1}x^n - U_n x^{n+2}\right)}{1 - V_1 x - x^2}$

Remark 8. *Horadam's theorem follows easily, since* $S_n = S^P_{n,1}(1)$. *Also* S_{-n} *can be found without difficulty, by observing that* $P_{-n} = pp_{-n-2} + qp_{-n-1} = -p(-1)^{n+2}p_{n+2} - q(-1)^{n+1}p_{n+1}$, *and using* $S^p_{n,1}(-1)$.

4 Weighted Combinatorial Sums

In [6] there are quite a few identities of the form $\sum_{i=0}^{n}\binom{n}{i}F_i = F_{2n}$, or $\sum_{i=0}^{n}\binom{n}{i}F_i^2$, which is $5^{[\frac{n-1}{2}]}L_n$ if n even, and $5^{[\frac{n-1}{2}]}F_n$, if n odd. A natural question is: *for fixed r, what is the closed form for the weighted sum $\sum_{i=0}^{n}\binom{n}{i}F_i^r$ (if it exists)?* We are able to answer the previous question, not only for the Fibonacci sequence, but also for any second-order recurrence sequences. Let $S_{r,n}(x)=\sum_{i=0}^{n}\binom{n}{i}U_i^r x^i$.

Theorem 9. *We have*

$$S_{r,n}(x)=\sum_{k=0}^{r}\binom{r}{k}A^k(-B)^{r-k}(1+\alpha^k\beta^{r-k}x)^n.$$

Moreover, if $U_0=0$, then $S_{r,n}(x)=A^r\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}(1+\alpha^k\beta^{r-k}x)^n$.

Proof. Let

$$\begin{aligned}
S_{r,n}(x) &= \sum_{i=0}^{n}\binom{n}{i}\sum_{k=0}^{r}\binom{r}{k}(A\alpha^i)^k(-B\beta^i)^{r-k}x^i\\
&= \sum_{k=0}^{r}\binom{r}{k}A^k(-B)^{r-k}\sum_{i=0}^{n}\binom{n}{i}(\alpha^k\beta^{r-k}x)^i\\
&= \sum_{k=0}^{r}\binom{r}{k}A^k(-B)^{r-k}(1+\alpha^k\beta^{r-k}x)^n
\end{aligned}$$

If $U_0=0$, then $A=B$, and $S_{r,n}(x)=A^r\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}(1+\alpha^k\beta^{r-k}x)^n$ □

Studying Theorem 9, we observe that we get nice sums involving the Fibonacci and Lucas sequences (or any such sequence, for that matter), if we are able to express 1 plus/minus a power of α,β as the same multiple of a power of α, respectively β. The following lemma turns out to be very useful.

Lemma 10. *The following identities are true*

$$\begin{aligned}\alpha^{2s}-(-1)^s =&\sqrt{5}\alpha^s F_s\\ \beta^{2s}-(-1)^s =&-\sqrt{5}\beta^s F_s\\ \alpha^{2s}+(-1)^s =&L_s\alpha^s\\ \beta^{2s}+(-1)^s =&L_s\beta^s.\end{aligned}\tag{6}$$

Proof. Straightforward using the Binet formula for F_s and L_s. $\square$

Theorem 11. *We have*

$$S_{4r,n}(1) = 5^{-2r}\left(\sum_{k=0}^{2r-1}(-1)^{k(n+1)}\binom{4r}{k}L_{2r-k}^n L_{(2r-k)n}+\binom{4r}{2r}2^n\right)\tag{7}$$

$$S_{4r+2,n}(1) = 5^{\frac{n+1}{2}-(2r+1)}\sum_{k=0}^{2r}\binom{4r+2}{k}F_{2r+1-k}^n F_{n(2r+1-k)},\ \textit{if } n \textit{ odd}\tag{8}$$

$$S_{4r+2,n}(1) = 5^{\frac{n}{2}-(2r+1)}\sum_{k=0}^{2r}(-1)^k\binom{4r+2}{k}F_{2r+1-k}^n L_{n(2r+1-k)}\ \textit{if } n \textit{ even.}\tag{9}$$

Proof. We use Theorem 9. Associating $k\leftrightarrow 4r+2-k$, except for the middle term in $S_{4r+2,n}(1)$, we obtain

$$\begin{aligned}S_{4r+2,n}(1) &= 5^{-(2r+1)}\sum_{k=0}^{2r}(-1)^k\binom{4r+2}{k}\left((1+\alpha^k\beta^{4r+2-k})^n+(1+\alpha^{4r+2-k}\beta^k)^n\right)\\ &= 5^{-(2r+1)}\sum_{k=0}^{2r}(-1)^k\binom{4r+2}{k}\left((1+(-1)^k\beta^{4r+2-2k})^n+(1+(-1)^k\alpha^{4r+2-2k})^n\right)\\ &= 5^{-(2r+1)}\sum_{k=0}^{2r}(-1)^{k(n+1)}\binom{4r+2}{k}\left(((-1)^k+\beta^{2(2r+1-k)})^n+((-1)^k+\alpha^{2(2r+1-k)})^n\right).\end{aligned}\tag{10}$$

We did not insert the middle term, since it is equal to

$$\begin{aligned}&5^{-(2r+1)}(-1)^{2r+1}\binom{4r+2}{2r+1}(1+\alpha^{2r+1}\beta^{2r+1})^n\\ &= 5^{-(2r+1)}(-1)^{2r+1}\binom{4r+2}{2r+1}(1+(-1)^{2r+1})^n = 0.\end{aligned}$$

Assume first that n is odd. Using (6) into (10), and observing that $\alpha^{2(2r+1-k)} - (-1)^{2r+1-k} = \alpha^{2(2r+1-k)} + (-1)^k$, we get

$$\begin{aligned}
S_{4r+2,n}(1) &= 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^{(n+1)k} \binom{4r+2}{k} 5^{\frac{n}{2}} F_{2r+1-k}^n \\
&\qquad \left((-1)^n \beta^{n(2r+1-k)} + \alpha^{n(2r+1-k)}\right) \\
&= 5^{-(2r+1)} \sum_{k=0}^{2r} \binom{4r+2}{k} 5^{\frac{n+1}{2}} F_{2r+1-k}^n F_{n(2r+1-k)}
\end{aligned}$$

Assume n even. As before,

$$\begin{aligned}
S_{4r+2,n}(1) &= 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^{(n+1)k} \binom{4r+2}{k} 5^{\frac{n}{2}} F_{2r+1-k}^n \\
&\qquad \left((-1)^n \beta^{n(2r+1-k)} + \alpha^{n(2r+1-k)}\right) \\
&= 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} 5^{\frac{n}{2}} F_{2r+1-k}^n L_{n(2r+1-k)}
\end{aligned}$$

In the same way, associating $k \leftrightarrow 4r-k$, except for the middle term,

$$\begin{aligned}
S_{4r,n}(1) &= 5^{-2r} \sum_{k=0}^{2r-1} (-1)^k \binom{4r}{k} \left((1+\alpha^k \beta^{4r-k})^n + (1+\alpha^{4r-k}\beta^k)^n\right) + 5^{-2r} \binom{4r}{2r} 2^n \\
&= 5^{-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} \left(\left((-1)^k + \beta^{2(2r-k)}\right)^n + \left((-1)^k + \alpha^{2(2r-k)}\right)^n\right) \\
&\qquad + 5^{-2r} \binom{4r}{2r} 2^n \qquad (11)\\
&= 5^{-2r} \left(\sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} \left(L_{2r-k}^n \beta^{(2r-k)n} + L_{2r-k}^n \alpha^{(2r-k)n}\right) + \binom{4r}{2r} 2^n\right) \\
&= 5^{-2r} \left(\sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} L_{2r-k}^n L_{(2r-k)n} + \binom{4r}{2r} 2^n\right).
\end{aligned}$$

□

Remark 12. *In the same manner we can find* $\sum_{i=0}^{n} \binom{n}{i} U_{pi}^r x^i$.

As a consequence of the previous theorem, for the even cases, and working out the details for the odd cases we get

Corollary 13. *We have*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{i} F_i = F_{2n} \\
&\sum_{k=0}^{2n} \binom{2n}{i} F_i^2 = 5^{n-1} L_{2n} \\
&\sum_{k=0}^{2n+1} \binom{2n+1}{i} F_i^2 = 5^n F_{2n+1} \\
&\sum_{k=0}^{n} \binom{n}{i} F_i^3 = \frac{1}{5}(2^n F_{2n} + 3F_n) \\
&\sum_{k=0}^{n} \binom{n}{i} F_i^4 = \frac{1}{25}(3^n L_{2n} - 4(-1)^n L_n + 6 \cdot 2^n).
\end{aligned}$$

Proof. The second, third and fifth identities follow from the previous theorem. Now, using Theorem 9, with $A = \frac{1}{\sqrt{5}}$, we get

$$\begin{aligned}
S_{1,n}(1) &= \frac{1}{\sqrt{5}} \sum_{k=0}^{1} (-1)^{1-k} \binom{1}{k} (1 + \alpha^k \beta^{1-k})^n \\
&= \frac{1}{\sqrt{5}}(-(1+\beta)^n + (1+\alpha)^n) = \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) = F_{2n}.
\end{aligned}$$

Furthermore, the fourth identity follows from

$$\begin{aligned}
S_{3,n}(1) &= \frac{1}{5\sqrt{5}} \sum_{k=0}^{3} (-1)^{3-k} \binom{3}{k} (1 + \alpha^k \beta^{3-k})^n \\
&= \frac{1}{5\sqrt{5}}(-(1+\beta^3)^n + 3(1+\alpha\beta^2)^n - 3(1+\alpha^2\beta)^n + (1+\alpha^3)^n) \\
&= \frac{1}{5\sqrt{5}}(-(2\beta^2)^n + 3\alpha^n - 3\beta^n + (2\alpha^2)^n) \\
&= \frac{1}{5}(2^n F_{2n} + 3F_n),
\end{aligned}$$

since $1 + \beta^3 = 2\beta^2$, $1 + \alpha^3 = 2\alpha^2$. ∎

We remark the following

Corollary 14. *We have, for any n,*

(i) $2^n F_{2n} + 3F_n \equiv 0 \pmod 5$

(ii) $3^n L_{2n} - 4(-1)^n L_n + 6 \cdot 2^n \equiv 0 \pmod{5^2}$

(iii) $\displaystyle\sum_{k=0}^{2r} \binom{4r+2}{k} F_{2r+1-k}^n F_{n(2r+1-k)} \equiv 0 \pmod{5^{4r+2-\frac{n-1}{2}}}$, *if n is odd, $n \le 8r+3$.*

(iv) $\displaystyle\sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} F_{2r+1-k}^n L_{n(2r+1-k)} \equiv 0 \pmod{5^{4r+2-\frac{n}{2}}}$, *if n is even, $n \le 8r+2$.*

(v) $\displaystyle\sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} L_{2r-k}^n L_{(2r-k)n} + \binom{4r}{2r} 2^n \equiv 0 \pmod{5^{2r}}$.

Taking other values for x (as desired) in Theorem 9, for instance, $x = -1$ and working out the details, we get the following

Theorem 15. *We have*

$$\begin{aligned}
S_{4r,n}(-1) &= 5^{\frac{n}{2}-2r} \sum_{k=0}^{2r-1} (-1)^k F_{2r-k}^n L_{(4r-2k)n} \binom{4r}{k}, \ \textit{if } n \textit{ even} \\
S_{4r,n}(-1) &= -5^{\frac{n+1}{2}-2r} \sum_{k=0}^{2r-1} F_{2r-k}^n F_{(4r-2k)n} \binom{4r}{k}, \ \textit{if } n \textit{ odd} \\
S_{4r+2,n}(-1) &= 5^{-(2r+1)} \left(\sum_{k=0}^{2r} (-1)^{k(n+1)+n} \binom{4r+2}{k} L_{2r+1-k}^n L_{(2r+1-k)n} - 2^n \binom{4r+2}{2r+1} \right).
\end{aligned}$$

Proof. We use $x = -1$ in Theorem 9. Associating $k \leftrightarrow 4r+2-k$ in $S_{4r+2,n}(-1)$, we obtain

$$\begin{aligned}
S_{4r+2,n}(-1) = 5^{-(2r+1)} &\sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} \left((1 - \alpha^k \beta^{4r+2-k})^n + (1 - \alpha^{4r+2-k} \beta^k)^n \right) \\
&- 5^{-(2r+1)} 2^n \binom{4r+2}{2r+1} \\
= 5^{-(2r+1)} &\sum_{k=0}^{2r} (-1)^k \binom{4r+2}{k} \left((1 - (-1)^k \beta^{4r+2-2k})^n + (1 - (-1)^k \alpha^{4r+2-2k})^n \right) \\
&- 5^{-(2r+1)} 2^n \binom{4r+2}{2r+1}
\end{aligned}$$

$$= 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^{k(n+1)} \binom{4r+2}{k} \Big(((-1)^k - \beta^{2(2r+1-k)})^n + ((-1)^k - \alpha^{2(2r+1-k)})^n \Big)$$

$$- 5^{-(2r+1)} 2^n \binom{4r+2}{2r+1}$$

$$= 5^{-(2r+1)} \sum_{k=0}^{2r} (-1)^{k(n+1)+n} \binom{4r+2}{k} L^n_{2r+1-k} L_{(2r+1-k)n} - 5^{-(2r+1)} 2^n \binom{4r+2}{2r+1},$$

since $(-1)^k - \beta^{4r+2-2k} = -L_{2r+1-k}\beta^{2r+1-k}$ and $(-1)^k - \alpha^{4r+2-2k} = -L_{2r+1-k}\alpha^{2r+1-k}$, by Lemma 10. In the same way, associating $k \leftrightarrow 4r-k$, with the middle term zero,

$$S_{4r,n}(1) = 5^{-2r} \sum_{k=0}^{2r-1} (-1)^k \binom{4r}{k} \Big((1 - \alpha^k \beta^{4r-k})^n + (1 - \alpha^{4r-k}\beta^k)^n \Big)$$

$$= 5^{-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} \left(\left((-1)^k - \beta^{2(2r-k)} \right)^n + \left((-1)^k - \alpha^{2(2r-k)} \right)^n \right)$$

$$= 5^{-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)} \binom{4r}{k} \left(5^{\frac{n}{2}} F^n_{2r-k} \beta^{(2r-k)n} + 5^{\frac{n}{2}} (-1)^n F^n_{2r-k} \alpha^{(2r-k)n} \right)$$

$$= 5^{\frac{n}{2}-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)+n} F^n_{2r-k} \binom{4r}{k} (\alpha^{(2r-k)n} + (-1)^n \beta^{(2r-k)n}),$$

since $(-1)^k - \beta^{4r-2k} = \sqrt{5} F_{2r-k}\beta^{2r-k}$ and $(-1)^k - \alpha^{4r-2k} = -\sqrt{5} F_{2r-k}\alpha^{2r-k}$, by Lemma 10. Therefore, for n even, $S_{4r,n}(1) = 5^{\frac{n}{2}-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)+n} F^n_{2r-k} L_{(4r-2k)n} \binom{4r}{k}$, and for n odd, $S_{4r,n}(1) = 5^{\frac{n+1}{2}-2r} \sum_{k=0}^{2r-1} (-1)^{k(n+1)+n} F^n_{2r-k} F_{(4r-2k)n} \binom{4r}{k}$. $\square$

A consequence for even powers and a similar idea for odd powers produces

Corollary 16. *We have*

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i\binom{n}{i}F_i &= -F_n\\
\sum_{i=0}^{n}(-1)^i\binom{n}{i}F_i^2 &= \frac{1}{5}\left((-1)^nL_n-2^{n+1}\right)\\
\sum_{i=0}^{n}(-1)^i\binom{n}{i}F_i^3 &= \frac{1}{5}\left((-2)^nF_n-3F_{2n}\right)\\
\sum_{i=0}^{n}(-1)^i\binom{n}{i}F_i^4 &= 5^{\frac{n}{2}-2}(L_{2n}-L_n),\ \text{if } n \text{ even}\\
\sum_{i=0}^{n}(-1)^i\binom{n}{i}F_i^4 &= -5^{\frac{n+1}{2}-2}(F_{2n}+4F_n),\ \text{if } n \text{ odd.}
\end{aligned}$$

Proof. The first identity is simple application of Theorem 9. The identities for even powers are consequences of Theorem 15. Now, using Theorem 9, we get

$$\begin{aligned}
S_{3,n}(-1) &= \frac{1}{5\sqrt{5}}\left(-(1-\beta^3)^n+3(1-\alpha\beta^2)^n-3(1-\alpha^2\beta)^n+(1-\alpha^3)^n\right)\\
&= \frac{1}{5\sqrt{5}}\left((-2)^n\beta^n+3\beta^{2n}-3\alpha^{2n}+(-2)^n\alpha^n\right) = \frac{1}{5}((-2)^nF_n-3F_{2n}),
\end{aligned}$$

since $1-\beta^3=-2\beta$, $1-\alpha^3=-2\alpha$. ☐

We remark the following

Corollary 17. *We have, for any n, $(-1)^nL_n-2^{n+1}\equiv 0 \pmod 5$ and $(-2)^nF_n-3F_{2n}\equiv 0 \pmod 5$.*

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